MATHEMATICS XII

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Topic

Matrices and Determinants

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Matrices and Determinants

Addition of Matrix: If $A=[a_{ij}]$,$B=[b_{ij}]$ be two matrices of same type m X n ,then their sum A+B is defined as the matrix $A+B= [a_{ij}+b_{ij}]$ or $C= [c_{ij}]$ where $c_{ij}=a_{ij}+b_{ij}$

Example

$$A = \begin{bmatrix} 2 & 0 & -9 \\ 0 & 7 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 9 & -3 \\ 0 & -5 & -7 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 2+1 & 0+9 & -9-3 \\ 0+0 & 7-5 & 3-7 \end{bmatrix} = \begin{bmatrix} 3 & 9 & -12 \\ 0 & 2 & -4 \end{bmatrix}$$

Matrices and Determinants

Subtraction of Matrix : If $A=[a_{ij}]$,$B=[b_{ij}]$ be two matrices of same type m X n ,then their difference A-B is defined as the matrix $A-B= [a_{ij}-b_{ij}]$ or $C= [c_{ij}]$ where $c_{ij}=a_{ij}-b_{ij}$

Example

$$A = \begin{bmatrix} 2 & 0 & -9 \\ 0 & 7 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 9 & -3 \\ 0 & -5 & -7 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 2-1 & 0-9 & -9+3 \\ 0-0 & 7+5 & 3+7 \end{bmatrix} = \begin{bmatrix} 1 & -9 & -6 \\ 0 & 12 & 10 \end{bmatrix}$$

Matrices Scalar Multiplication

If each element of a matrix A= $[a_{ij}]$ is multiplied by a scalar k then resulting matrix kA= $[ka_{ij}]$

Example

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 4 & 5 & -1 \\ 6 & 0 & 2 \end{bmatrix}$$

$$3A = \begin{bmatrix} 3 & -6 & 9 \\ 12 & 15 & -3 \\ 18 & 0 & 6 \end{bmatrix}$$

Product of Matrices

Let $A=[a_{ij}]$, $B=[b_{ij}]$ be two matrices of type m X n and n X p . (The number of columns of A is the same as the number of rows of B)

Then product of A and B is $C=AB = [c_{ik}]_{mp}$

where

$$c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$$

where $1 \le i \le m, 1 \le j \le n, 1 \le k \le p$

Example of product of Matrices

$$A = \begin{bmatrix} 2 & 5 \\ 6 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2 & 5 \\ 6 & 3 \end{bmatrix} \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2*2+5*5 & 2*3+5*6 & 2*4+5*7 \\ 6*2+3*5 & 6*3+3*6 & 6*4+3*7 \end{bmatrix}$$

$$= \begin{bmatrix} 4+25 & 6+30 & 8+35 \\ 12+15 & 18+18 & 24+21 \end{bmatrix}$$

$$= \begin{bmatrix} 29 & 36 & 43 \\ 27 & 36 & 45 \end{bmatrix}$$

Properties of product of Matrices

1. Matrices multiplication is not commutative.
2. Matrices multiplication is associative.
3. Matrices multiplication is left or right distributive.

Transpose of a Matrices

The matrix obtained from a given matrix A ,by interchanging its rows and columns, is called the transpose of A and is denoted A' or A^t or A^T

Example $A = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix}$

Then Transpose of A= A'= $\begin{bmatrix} 2 & 5 \\ 3 & 6 \\ 4 & 7 \end{bmatrix}$

Reversal Law for Matrices

If A and B are any two matrices conformable for multiplication , then (AB)’ = B’A’.

Symmetric Matrix– Any square matrix $A=[a_{ij}]$ is said to be symmetric if $a_{ij}=a_{ji}$ or A’=A .

Skew Symmetric Matrix-- Any square matrix $A=[a_{ij}]$ is said to be skew symmetric if $a_{ij}=-a_{ji}$ or A’=-A .

Invertible Matrix, Inverse of a matrix

If A is any square matrix of order n and there exist another square matrix B of the same order n, such that AB=BA =I , then B is called an inverse matrix of A and is denoted by A^{-1} .

Thus B is inverse of A or $B=A^{-1}$.

Elementary Operation on a matrix

1. The interchange of two rows or two columns.
2. The multiplication of the elements of any row or column by a non zero number.
3. The addition of the elements of any row or column, the corresponding elements of any other row or column multiplied by any non zero number.

Symbols (1) $R_1 \leftrightarrow R_2$ (2) $R_1 \rightarrow kR_1$ (3) $R_1 \rightarrow R_1 + kR_2$

Inverse of a Matrix by elementary operations

Two Important results:-

1. Every elementary row transformation of a product $X=AB$ can be obtained by subjecting the pre-factor to the same elementary row transformation.

2. Every elementary column transformation of a product $X=AB$ can be obtained by subjecting the post factor to the same elementary column transformation.

Inverse of a Matrix by elementary operations

Method

1. Write A=IA and go on performing elementary row transformation on the product and pre-factor of A till we reach the result I=BA . Then B is the inverse of A.

2. Write A=AI and go on performing elementary column transformation on the product and post factor of A till we reach the result I=AB . Then B is the inverse of A.

Important note on inverse

Note

In case after applying one or more elementary row (column) transformation on A=IA (A=AI) , if we obtain all zeros in one or more rows (Columns) of the matrix A on L.H.S. , then A^{-1} does not exist.

Inverse of a Matrix by elementary operations

By using elementary row transformation, find the inverse of matrix .

1. $$A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$$

2. $$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$$

Determinants

Def. With every square matrix $A=[a_{ij}]$ we associate a number called determinant of A and is denoted by det A or I A I

The determinant of a 1 X 1 matrix $[a_{11}]$ is defined to be a_{11}

The determinant of a 2 X 2 matrix

$$\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

Determinants

The determinant of a 3 X 3 matrix

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32})$$
$$- a_{12}(a_{21}a_{33} - a_{31}a_{23})$$
$$+ a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

To Evaluate Determinants

(1) The determinant of a 1X1 matrix $A=[a_{11}]$ is defined to be a_{11} i.e. $|A|=a_{11}$

(2) The determinant of a 2X2 matrix

$$A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$ is $= a_{11}.a_{22} - a_{21}.a_{12}$

Determinants

Let A=[a_{ij}] be a square matrix of order n , then

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & .. & a_{2n} \\ a_{31} & a_{32} & a_{33} & .. & a_{3n} \\ .. & .. & .. & .. & .. \\ a_{n1} & a_{n2} & a_{n3} & .. & a_{nn} \end{vmatrix} \quad \text{or} \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & .. & a_{2n} \\ a_{31} & a_{32} & a_{33} & .. & a_{3n} \\ .. & .. & .. & .. & .. \\ a_{n1} & a_{n2} & a_{n3} & .. & a_{nn} \end{vmatrix}$$

Is associated to the square matrix A and is called determinant of matrix A denoted as I A I or det A.

To Evaluate Determinants

Let A= $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ Expanding by first row

$=(-1)^{1+1}a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2}\, a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

$+(-1)^{1+3}\, a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

$=a_{11}(a_{22}.a_{33}-a_{23}.a_{32}) - a_{12}(a_{21}.a_{33} - a_{31}.a_{23})$

$+ a_{13}(a_{21}.a_{32} - a_{31}.a_{22})$

To Evaluate Determinants

Let A= $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ Expanding by second row

$$=(-1)^{2+1}a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} +(-1)^{2+2}a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

$$+(-1)^{2+3}a_{23}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

$$=-a_{21}(a_{12}.a_{33}-a_{13}.a_{32}) + a_{22}(a_{11}.a_{33} - a_{31}.a_{13})$$

$$- a_{23}(a_{11}.a_{32} - a_{31}.a_{12})$$

Notations

Let Δ be the given determinant. Then

(i) R_1, R_2, R_3 stand for first, second and third rows of Δ.

(ii) C_1, C_2, C_3 stand for first, second and third columns of Δ .

(iii) By $R_2 \rightarrow R_2 - R_3$ we mean that third row is to be subtracted from 2nd row.

(iv) By $C_1 \rightarrow C_1 + 2C_2 - 3C_3$, we mean that we are to add in first column, the two times of C_2 and subtract three times C_3.

Properties of Determinants

Property 1. If each element of a row (column)

of a determinant is zero , then value of determinant is zero.

Property 2. Value of a determinant is not changed by changing the rows into columns and columns into rows.

Properties of Determinants

Property 3. If two adjacent rows (columns)of a determinant are interchanged , then the sign of the determinant is changed but its numerical value is unchanged.

Property 4. If two rows (columns) are identical, then the value of the determinant is zero.

Property 5. If every element of a row (column) is multiplied by some constant k, the value of the determinant is multiplied by

Properties of Determinants

Property 6 .If each element in any row (column) consist of two terms , then the determinant can be expressed as the sum of the determinants of same order.

Property 7 . The value of a determinant remain unchanged if to each element of a row (column) be add (or subtracted) equimultiplies of the corresponding elements of one or more rows (columns) of the determinant.

Properties of Determinants

Property 8. The value of the determinant of a diagonal matrix is equal to the product of the diagonal elements.

Property 9. The value of the determinant of a skew-symmetric matrix of odd order is always zero.

Property 10. The determinant of a symmetric matrix of even order is always a perfect square.

Board Questions

1. Using properties of determinants ,prove that

$$\begin{vmatrix} x-y-z & 2x & 2x \\ 2y & y-z-x & 2y \\ 2z & 2z & z-x-y \end{vmatrix} = (x+y+z)^3$$

mar. 03 (B)

2. Using properties of determinants ,prove that

$$\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ca \\ 1 & c & c^2-ab \end{vmatrix} = 0$$

Board Questions

3. If A+B+C=π then find the value of

$$\begin{vmatrix} \sin(A+B+C) & \sin B & \cos C \\ -\sin B & 0 & \tan A \\ \cos(A+B) & -\tan A & 0 \end{vmatrix}$$

March 06 (C)

4.Prove;

$$\begin{vmatrix} a & b-c & c-b \\ a-c & b & c-a \\ a-b & b-a & c \end{vmatrix} = (a+b-c)(b+c-a)(c+a-b)$$

July 08 (C)

Example

Prove that

$$\begin{vmatrix} x+a & b & c \\ a & x+b & c \\ a & b & x+c \end{vmatrix} = x^2(x+a+b+c)$$

Using properties of determinants ,prove that

$$\begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix} = (x-y)(y-z)(z-x)(xy+yz+zx)$$

Example

Without expanding prove that

$$\begin{vmatrix} 9 & 9 & 12 \\ 1 & -3 & -4 \\ 1 & 9 & 12 \end{vmatrix} = 0$$

Using properties of determinants ,prove that following determinant vanish

$$\begin{vmatrix} 43 & 1 & 6 \\ 35 & 7 & 4 \\ 17 & 3 & 2 \end{vmatrix}$$

Example

Without expanding prove that

$$\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

Using properties of determinants ,prove that following determinant vanish

$$\begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix}$$

Applications of Determinants

1. Area of a Triangle The area of a triangle with vertices (x_1,y_1), (x_2,y_2), (x_3,y_3) is given by

$$\Delta=1/2\ [\ x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)]$$

In determinant form

$$\Delta=\ \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Because area is always positive so we take absolute value of determinant.

Minors and Co-factors

Minor : Minor of an element A_{ij} of a determinant is the determinant obtained by deleting its i^{th} row and j^{th} column in which element a_{ij} lies. Minor of an element a_{ij} is denoted by M_{ij}.

Co-factor : Cofactor of an element A_{ij} , denoted by A_{ij} is defined by $A_{ij} = (-1)^{i+j} M_{ij}$

Adjoint of a square matrix

Let $A=[a_{ij}]$ be any square matrix.

The transpose of the matrix $[A_{ij}]$, where A_{ij} denotes the cofators of a_{ij} in I A I, is called adjoint of A and is denoted by adj.A.

Singular Matrix : Any matrix whose determinant is zero, is singular matrix.

Non-singular Matrix : Any matrix whose determinant is not zero, is non-singular matrix.

Inverse of a matrix

If A is any square matrix of order n and there exist another square matrix B of the same order n, such that AB=BA =I , then B is called an inverse matrix of A and is denoted by A^{-1}

.

Inverse of A= $A^{-1} = \frac{\text{Adj (A)}}{|A|}$

Consistent and Inconsistent Solutions

Consistent system : A system of equation is said to be consistent if its solution (one or more) exists.

Inconsistent system : A system of equation is said to be inconsistent if its solution does not exist.

Working rule to check consistency

Case I When $|A| \neq 0$

System is consistence and has unique solution.

Case II When $|A| = 0$.

Find Adj(A) and then find Adj(A) .B

If Adj(A) .B $\neq 0$ then system is inconsistence .

Case III If Adj(A).B=0 Then it may have infinite solutions then it is consistence or have no solution then it is inconsistence.

Solution of linear equations

Examine the consistencies of the system of equations

1. $3x-y+2z=3$

   $2x+y+3z=5$

   $x-2y-z=1$

2. $x-y+z=3$

   $2x+y-z=2$

   $-x-2y+2z=1$

Solution of linear equations

Solve the equations by matrix method :

1. $5x+2y=4$

   $7x+3y=5$

2. $x+2y-3z=6$

   $3x+2y-2z=3$

   $2x-y+z=2$

Solution of linear equations

Solve the equations by matrix method :

$$2x-3y+5z=16$$

$$3x+2y-4z=-4$$

$$x+y-2z=-3$$

Solution of linear equations

If $A = \begin{vmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{vmatrix}$, find A^{-1}

Using A^{-1} ,Solve the equations

$$2x-3y+5z=11$$

$$3x+2y-4z=-5$$

$$x+ y- 2z=-3$$

Question

Compute $(AB)^{-1}$ where

$$A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

Question

If x,y,z are non zero real numbers, then the inverse of matrix A is

$$A = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$$

$$(A) = \begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$$

$$(\mathrm{B}) = (xyz)\begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$$

$$(C) = \frac{1}{xyz}\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$$

$$(D) = \frac{1}{xyz}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thanks